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IMPULSIVE LOADING OF IDEAL FIBRE-REINFORCED RIGID-PLASTIC BEAMS—I
FREE BEAM UNDER CENTRAL IMPACT

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Abstract—A theory was developed in [1] for the dynamical behaviour under transverse load of ideal fibre-reinforced beams (that is, beams which are inextensible in their longitudinal direction) which exhibit rigid-plastic mechanical response. This theory is here applied to the problem of a beam of finite length, free at both ends, which is struck centrally by a mass which subsequently adheres to the beam. The general solution for the motion of the beam is determined for a fairly wide class of non-linear strain-hardening laws. Simplified approximate solutions are derived for the cases of (a) a heavy striker, (b) a light striker and (c) low impact speed and/or slight strain-hardening.

1. INTRODUCTION

The general theory of the behaviour of ideal fibre-reinforced rigid-plastic beams was developed in [1]. General accounts of the theory of ideal fibre-reinforced materials have been given by Spencer[2], Pipkin[3] and Rogers[4]. The term 'ideal fibre-reinforced beam' is used here to describe a beam which is inextensible in its axial direction. In the context of a rigid-plastic theory such a beam may be regarded as the limiting case of a beam made of material whose shear yield stress (for shear on planes parallel to the beam axis) is much less than its tensile yield stress in the axial direction. For example, the beam may consist of a ductile metallic matrix reinforced in the axial direction by strong stiff fibres, or be of laminated or sandwich construction in which the laminations are alternately strong and ductile. A beam of this type will tend to deform by shear rather than by flexure; in the limit of an ideal fibre-reinforced beam, shear is the only allowed deformation mechanism.

Under certain assumptions stated in [1], it was shown there that discontinuities in slope and velocity may propagate along an ideal fibre-reinforced beam of strain-hardening rigid-plastic material. The theory was applied to the problem of a beam, moving normal to its axis, whose mid-point is suddenly brought to rest by transverse impact on a rigid stop. Explicit solutions to the problem were obtained for the case of linear strain-hardening in [1], and subsequently extended to a more general strain-hardening law in [5]. Some further solutions, for the linear strain-hardening case, have been given by Jones[6]. In this series of papers, we give some further solutions and investigate the effect of various parameters on these solutions. These solutions are all for impulsive loading of a beam struck transversely by a mass which subsequently adheres to the beam. In Part I we consider a free beam struck at its mid-point; in Part II we consider a beam supported at its ends and struck at any point; and in Part III we consider a cantilever beam struck at any point. A discussion of the solutions, in the light of the approximations made in formulating the theory, is given at the end of Part III.

2. SUMMARY OF GENERAL THEORY

A full account of the theory is given in [1]. The notation used here differs a little from that employed in [1]. Time is denoted by T and distance along the beam in its initial position by X. The deflection of the beam (assumed small) is denoted by $U(X, T)$ and its velocity in the transverse direction by $V(X, T)$. The slope of the beam is $\gamma(X, T)$, so that

$$\gamma(X, T) = \frac{\partial U(X, T)}{\partial X}, \quad V(X, T) = \frac{\partial U(X, T)}{\partial T}. \tag{2.1}$$

If V and γ are discontinuous at $X = A(T)$, then continuity of displacement at $X = A(T)$ gives the kinematic condition

$$\frac{\mathrm{d}A}{\mathrm{d}T}[\gamma] = -[V], \tag{2.2}$$

where, for any function $\phi(X, T)$, $[\phi]$ denotes the jump in ϕ across $X = A(T)$. Thus

$$[\phi] = \phi(A+0, T) - \phi(A-0, T). \tag{2.3}$$

The shear force on the cross-section X at time T is denoted $Q(X, T)$. The yield condition takes the form

$$|Q| \leq Q_p, \tag{2.4}$$

where the yield shear force Q_p depends on the history of γ. For simplicity we assume that γ is, at each X, either a non-increasing or non-decreasing function of T, and then we may take

$$Q_p = Q_p(|\gamma|), \tag{2.5}$$

where Q_p is an increasing function of $|\gamma|$. The equality sign holds in (2.4) whenever deformation is taking place; in particular it applies at a section across which γ undergoes a discontinuous change. Thus if Q_p is a continuous function of $|\gamma|$, then Q is discontinuous at $X = A(T)$. Conservation of momentum of a beam element crossed by the discontinuity gives the dynamic jump condition

$$m\frac{\mathrm{d}A}{\mathrm{d}T}[V] = -[Q], \tag{2.6}$$

where m is the mass of the beam per unit length, and is assumed to be constant. We make the usual assumption that the material entering a plastic region is about to yield, so that $Q = Q_p$ at $X = A+0$ and $X = A-0$, and (2.6) takes the form

$$m\frac{\mathrm{d}A}{\mathrm{d}T}[V] = \mp[Q_p] \tag{2.7}$$

where the upper sign holds if Q is positive and the lower if Q is negative. For positive plastic working it is necessary that $Q\partial\gamma/\partial t > 0$, which implies $Q\gamma > 0$ since γ is monotonic.

In order to obtain explicit results, we shall use the special strain-hardening relation

$$Q_p = Q_0 + Q_1|\gamma|^n, \tag{2.8}$$

where Q_0, Q_1 and n are positive constants. Then (2.7) becomes

$$m\frac{\mathrm{d}A}{\mathrm{d}T}[V] = \mp Q_1[|\gamma|^n]. \tag{2.9}$$

In particular, if $n = 1$ we have the case of linear strain-hardening. Then (2.9) becomes

$$m\frac{\mathrm{d}A}{\mathrm{d}T}[V] = \mp Q_1[|\gamma|] \tag{2.10}$$

and (2.2) and (2.10) give

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \pm c, \quad \text{where} \quad mc^2 = Q_1. \tag{2.11}$$

However, we shall often consider the more general case in which n is not restricted to have the value one.

Segments of the beam may move as rigid bodies. In such segments, or in any region where Q, V and γ are continuous, we have

$$\frac{\partial Q}{\partial X}+P(X,T)=m\frac{\partial V}{\partial T}, \tag{2.12}$$

where $P(X, T)$ is the resultant force in the positive Y-direction applied, per unit length, to the beam.

A feature of problems involving ideal fibre-reinforced materials is the occurrence of singular surfaces (sheets of fibres) which carry infinite stress but finite force. For a rectangular beam $0 \leq Y \leq H$, of cross-sectional area S, the surfaces $Y=H$, $Y=0$ are singular, and the tensile force $F(X, T)$ per unit length in the Z-direction in these surfaces is given by

$$\frac{\partial F}{\partial X}=\pm\frac{Q}{S} \tag{2.13}$$

where the positive sign applies at $Y=H$ and the negative sign at $Y=0$.

We suppose that the beam initially lies along the X-axis from $X=-L$ to $X=L$ and is struck at time $T=0$ by a mass $2M$ travelling with speed V_0. It is convenient to introduce the non-dimensional variables

$$x=\frac{X}{L},\quad a=\frac{A}{L},\quad u=\frac{U}{L},\quad v=\frac{V}{V_0},\quad t=\frac{TV_0}{L}. \tag{2.14}$$

Then

$$\gamma=\frac{\partial u}{\partial x},\quad v=\frac{\partial u}{\partial t},\quad \frac{\mathrm{d}A}{\mathrm{d}T}=V_0\frac{\mathrm{d}a}{\mathrm{d}t}=V_0\dot{a}, \tag{2.15}$$

where, here and henceforth, superposed dots denote differentiation with respect to t. We also introduce the non-dimensional parameters

$$\alpha=\frac{M}{mL},\quad \beta^2=\frac{mV_0^2}{Q_0},\quad \omega^{2n}=\frac{Q_1}{Q_0}. \tag{2.16}$$

In terms of these non-dimensional variables and constants, eqns (2.2) and (2.9) become

$$\dot{a}[\gamma]=-[v], \tag{2.17}$$

$$\dot{a}[v]=\mp\omega^{2n}\beta^{-2}[|\gamma|^n]. \tag{2.18}$$

The parameters α, β, ω and n can be given physical interpretations. The parameter α is the ratio of the mass of the striker to the mass of the beam. We shall use the terms 'heavy striker' for the case $\alpha \gg 1$, and 'light striker' for the case $\alpha \ll 1$. The parameter β is related to the impact velocity and the speed of propagation of a discontinuity. If we denote (as in (2.11)) $c^2=Q_1/m$, then from (2.2) and (2.9)

$$\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right)^2=\pm c^2\frac{[|\gamma|^n]}{[\gamma]}. \tag{2.19}$$

As noted above, this reduces to $\mathrm{d}A/\mathrm{d}T=\pm c$ in the case $n=1$. In general

$$\frac{V_0}{c}=V_0\left(\frac{m}{Q_1}\right)^{1/2}=\omega^{-n}\beta, \tag{2.20}$$

so $\omega^{-n}\beta$ relates the impact speed to the propagation speed. The parameter ω is a measure of the rate of strain-hardening; the limit $\omega \to 0$ corresponds to a perfectly-plastic (non-hardening) material with yield shear force Q_0. The constant n is available (with Q_0 and Q_1) for fitting empirical shear force-deflection gradient curves; Fig. 1 illustrates the shapes of these curves for $n = 0$, $0 < n < 1$, $n = 1$ and $n > 1$. For algebraic convenience we also introduce a constant q defined as

$$q = \frac{2n}{n+1}, \quad n = \frac{q}{2-q}, \tag{2.21}$$

and note that q increases from 0 to 2 as n increases from 0 to ∞, and $q = 1$ when $n = 1$. The limit $q \to 0$ or $n \to 0$ gives a perfectly-plastic material with yield shear force $Q_0 + Q_1$.

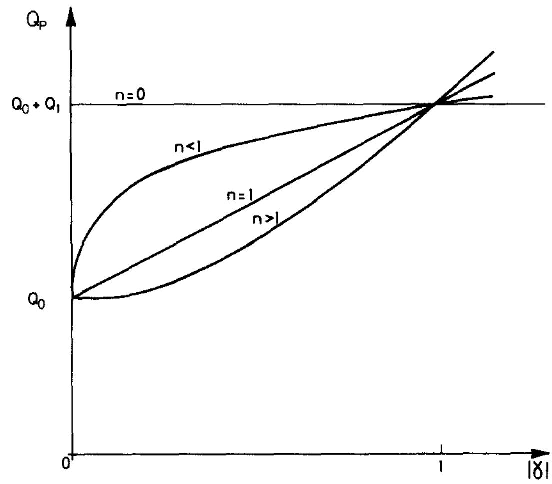

Fig. 1. The strain-hardening relation $Q_p = Q_0 + Q_1|\gamma|^n$.

The momentum $2I$ and kinetic energy $2E$ of the striker before impact are, respectively,

$$2I = 2MV_0, \quad 2E = MV_0^2, \tag{2.22}$$

and we note the relations

$$\alpha\beta = \frac{I}{L(mQ_0)^{1/2}}, \quad \alpha\beta^2 = \frac{2E}{Q_0 L}. \tag{2.23}$$

3. CENTRAL IMPACT OF FREE BEAM. FORMULATION

In this section we consider that the beam is free to move and initially lies along the X-axis from $X = -L$ to $X = L$. It is struck at time $T = 0$ by the mass $2M$ moving with speed V_0 in the positive Y-direction, and the mass adheres to the beam after impact. We seek solutions in which, during the subsequent motion, the configuration of the beam is of the form illustrated in Fig. 2. It is sufficient to consider the right-hand half $0 \leq X \leq L$ of the beam. In this CP moves as a rigid body with speed V_1 and PR moves as a rigid body with speed V_2. V and γ are discontinuous at time T at the point $X = A(T)$. The value of γ in the deformed segment $0 < X < A(T)$ is denoted by $f(X/L)$, so that

$$\gamma = \begin{cases} f(X/L), & 0 < X < A(T), \\ -f(-X/L), & -A(T) < X < 0, \\ 0, & -L < X < -A(T) \text{ and } A(T) < X < L. \end{cases} \tag{3.1}$$

The initial conditions are $V_1 = V_0$, $V_2 = 0$, $A = 0$ when $T = 0$. We note that $\gamma \leq 0$ in $0 < X < L$,

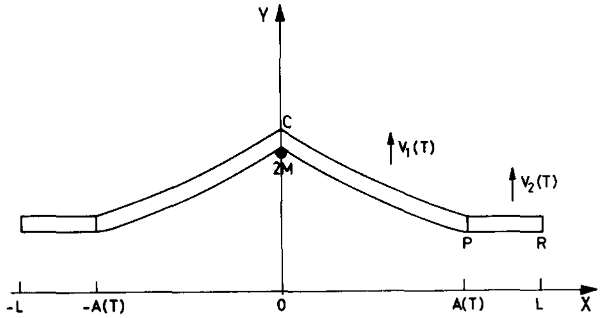

Fig. 2. Central impact of a free beam. Assumed form of deformation.

so in this region

$$Q_p = Q_0 + Q_1(-\gamma)^n. \tag{3.2}$$

The governing equations are as follows:
(a) Equation of motion of CP:

$$(M + mA)\frac{dV_1}{dT} = -Q_0 - Q_1\left\{-f\left(\frac{A}{L}\right)\right\}^n. \tag{3.3}$$

(b) Equation of motion of PR:

$$m(L-A)\frac{dV_2}{dT} = Q_0. \tag{3.4}$$

(c) Dynamic jump condition at $X = A$ (from (2.9)):

$$m\frac{dA}{dT}(V_1 - V_2) = Q_1\left\{-f\left(\frac{A}{L}\right)\right\}^n. \tag{3.5}$$

(d) Kinematic jump condition at $X = A$ (from (2.2)):

$$V_1 - V_2 = -\frac{dA}{dT}f\left(\frac{A}{L}\right). \tag{3.6}$$

We introduce the non-dimensional variables (2.14) and non-dimensional parameters (2.16), together with

$$v_1 = \frac{V_1}{V_0}, \quad v_2 = \frac{V_2}{V_0}. \tag{3.7}$$

In terms of these, eqns (3.3) to (3.6) become

$$\beta^2(\alpha + a)\dot{v}_1 = -1 - \omega^{2n}\{-f(a)\}^n, \tag{3.8}$$

$$\beta^2(1-a)\dot{v}_2 = 1, \tag{3.9}$$

$$\beta^2\dot{a}(v_1 - v_2) = \omega^{2n}\{-f(a)\}^n, \tag{3.10}$$

$$v_1 - v_2 = -\dot{a}f(a), \tag{3.11}$$

and the initial conditions become

$$v_1 = 1, \quad v_2 = 0, \quad a = 0, \quad \text{when } t = 0. \tag{3.12}$$

Equations (3.10) and (3.11) then give the initial values of $\dot{a}$ and γ as

$$\dot{a}(0) = (\beta^2\omega^{-2n})^{-1/(n+1)}, \quad f(0) = -(\beta^2\omega^{-2n})^{1/(n+1)}. \tag{3.13}$$

This value of $f(0)$ gives the slope of the beam at $x = 0$ after the impact. We note that for given properties m, Q_0, Q_1 of the beam, $f(0)$ depends only on the speed V_0 of the striker and not on its mass M, and that $f(0) \to \infty$ as $V_0 \to \infty$.

4. GENERAL SOLUTION

By adding (3.8), (3.9) and (3.10), integrating, and inserting the initial conditions, we obtain

$$(a + \alpha)v_1 + (1 - a)v_2 = \alpha. \tag{4.1}$$

This equation expresses conservation of linear momentum for the entire beam and striker. From (3.10), (3.11) and (4.1) there follows, by eliminating $f(a)$ and v_1,

$$\beta^2\dot{a}^{n+1} = \omega^{2n}\left\{\frac{\alpha - (1+\alpha)v_2}{a+\alpha}\right\}^{n-1}. \tag{4.2}$$

Then, by eliminating t from (3.9) and (4.2),

$$\frac{da}{dv_2} = (\omega\beta)^q(1-a)\left\{\frac{\alpha - (1+\alpha)v_2}{a+\alpha}\right\}^{q-1}, \tag{4.3}$$

where q is given by (2.21).

By integrating (4.3) and inserting the initial conditions, there follows

$$\left(\frac{\alpha}{1+\alpha} - v_2\right)^q = \left(\frac{\alpha}{1+\alpha}\right)^q - q(\omega\beta)^{-q}\left\{F_{q-1}\left(\frac{a+\alpha}{1+\alpha}\right) - F_{q-1}\left(\frac{\alpha}{1+\alpha}\right)\right\}, \tag{4.4}$$

where

$$F_{q-1}(z) = \int_0^z \frac{\zeta^{q-1}\,d\zeta}{1-\zeta} = z^q\sum_{r=0}^{\infty}\frac{z^r}{q+r} \qquad (0 \leq z < 1). \tag{4.5}$$

The function $F_{q-1}(z)$ is a special case of an incomplete beta function (see, for example, Abramowitz and Stegun[7]). Equation (4.4) determines v_2 in terms of $a(t)$. Equation (4.1) then gives

$$\left(v_1 - \frac{\alpha}{1+\alpha}\right)^q = \left(\frac{1-a}{a+\alpha}\right)^q\left\{\left(\frac{\alpha}{1+\alpha}\right)^q - q(\omega\beta)^{-q}\left[F_{q-1}\left(\frac{a+\alpha}{1+\alpha}\right) - F_{q-1}\left(\frac{\alpha}{1+\alpha}\right)\right]\right\}, \tag{4.6}$$

which determines v_1 in terms of $a(t)$.

The beam ceases to deform when $v_1 = v_2$. The discontinuity then stops propagating and the beam subsequently moves as a rigid body with speed $v_f V_0$, where, from (4.1),

$$v_f = \frac{\alpha}{1+\alpha}. \tag{4.7}$$

Hence (4.4) and (4.6) may be written

$$(v_f - v_2)^q = v_f^q - q(\omega\beta)^{-q}\left\{F_{q-1}\left(v_f + \frac{a}{1+\alpha}\right) - F_{q-1}(v_f)\right\}, \tag{4.8}$$

$$(v_1 - v_f)^q = \left(\frac{1-a}{a+\alpha}\right)^q\left[v_f^q - q(\omega\beta)^{-q}\left\{F_{q-1}\left(v_f + \frac{a}{1+\alpha}\right) - F_{q-1}(v_f)\right\}\right]. \tag{4.9}$$

The final value a_f of a is therefore the root of

$$F_{q-1}\left(v_f+\frac{a}{1+\alpha}\right)=F_{q-1}(v_f)+q^{-1}v_f^{\,q}(\omega\beta)^q. \tag{4.10}$$

It is evident from (4.5) and (4.7) that this equation has a single root between $a=0$ and $a=1$, and so the discontinuity always comes to rest before it reaches the end of the beam.

From (3.10) and (3.11)

$$f(a)=-\omega^{-q}\beta^{2-q}(v_1-v_2)^{2-q},$$

and hence, from (4.8) and (4.9),

$$f(a)=-\beta^2(\omega\beta)^{-q}\left(v_f+\frac{a}{1+\alpha}\right)^{q-2}\left[v_f^{\,q}-q(\omega\beta)^{-q}\left\{F_{q-1}\left(v_f+\frac{a}{1+\alpha}\right)-F_{q-1}(v_f)\right\}\right]^{(2-q)/q}. \tag{4.11}$$

Now substituting from (4.8), (4.9) and (4.11) into (3.11) and integrating gives

$$t=\beta^2(\omega\beta)^{-q}\int_0^a\left(v_f+\frac{a}{1+\alpha}\right)^{q-1}\left[v_f^{\,q}-q(\omega\beta)^{-q}\left\{F_{q-1}\left(v_f+\frac{a}{1+\alpha}\right)-F_{q-1}(v_f)\right\}\right]^{(1-q)/q}\mathrm{d}a, \tag{4.12}$$

and this determines $a(t)$ implicitly. The deflection $Lu(x,t)$ of the beam at time t, relative to its end points, is given by

$$u(x,t)=\begin{cases}-\displaystyle\int_x^{a(t)}f(x)\,\mathrm{d}x, & 0\leqslant x\leqslant a(t),\\ 0, & a(t)\leqslant x\leqslant 1.\end{cases} \tag{4.13}$$

The shear force in the rigid sections of the beam and the tensions in the singular fibres are readily determined from (2.12) and (2.13). It may be verified that the rate of plastic working at each shear hinge is positive, and that the yield condition is not violated in the rigid segments.

The above results simplify considerably for the case of linear hardening, $n=1$, $q=1$. Then $F_{q-1}(z)$ reduces to $-\log(1-z)$ and, for example, an explicit value of a_f can be obtained by solving (4.10). This linear hardening case has been solved by Jones[6] and so is not discussed further.

The function $F_{q-1}(z)$ can also be expressed in terms of standard functions for certain other values of n. For example, for $q=3/2$ ($n=3$), we have

$$F_{1/2}(z)=-2\sqrt{z}+\log\frac{1+\sqrt{z}}{1-\sqrt{z}},$$

and for $q=1/2$ ($n=1/3$), we have

$$F_{-1/2}(z)=\log\frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

5. HEAVY STRIKER: $\alpha\to\infty$

From (4.5) we have

$$F'_{q-1}(v_f)=\frac{v_f^{\,q-1}}{1-v_f}=(1+\alpha)\left(\frac{\alpha}{1+\alpha}\right)^{q-1}.$$

Hence

$$F_{q-1}\left(v_f+\frac{a}{1+\alpha}\right)-F_{q-1}(v_f)=\frac{a}{1+\alpha}F'_{q-1}(v_f)+O\left\{\left(\frac{a}{1+\alpha}\right)^2\right\}\to a\quad\text{as }\alpha\to\infty.$$

The values of v_2, v_1, γ and t in this limit are then readily obtained from (4.8), (4.9), (4.11) and (4.12). In this case the problem is identical to the one discussed in [5] except that the direction of the Y-axis is reversed and a uniform speed V_0 is superposed on the solution given in [5]. The results of [5] may be recovered by taking the limit $\alpha \to \infty$.

6. LIGHT STRIKER: $\alpha \ll 1$

The solution to be given in this section is valid for $\alpha \ll 1$ provided that $\omega\beta$ and q^{-1} are not large compared to one. It is also valid, without restriction on α, if $\omega\beta \ll 1$. However, it will be shown in Section 7 that when $\omega\beta \ll 1$, further simplification is possible.

We note first that if $a \ll 1$ (with no restriction on α) we have, from (4.5),

$$F_{q-1}\left(v_f + \frac{a}{1+\alpha}\right) - F_{q-1}(v_f) = \frac{a}{1+\alpha} F'_{q-1}(v_f) + O(a^2) = a v_f^{q-1} + O(a^2). \tag{6.1}$$

Thus if $a_f \ll 1$, the solution of (4.10) is, to first order,

$$a_f = q^{-1} v_f (\omega\beta)^q. \tag{6.2}$$

Since $\alpha \ll 1$ implies $v_f \ll 1$, it follows that $\alpha \ll 1$ implies $a_f \ll 1$ provided that $\omega\beta$ and q^{-1} are not large compared to one. Then, since a_f is the maximum value of a, an approximate solution for small α is given by neglecting the terms of order a^2 in (6.1). To this approximation, (4.8) and (4.9) become

$$\begin{aligned} \left(1 - \frac{v_2}{v_f}\right)^q &= 1 - \frac{a}{a_f}, \\ \left(\frac{v_1}{v_f} - 1\right)^q &= (a+\alpha)^{-q}\left(1 - \frac{a}{a_f}\right). \end{aligned} \tag{6.3}$$

Also, from (4.11),

$$f(a) = -\beta^2(\omega\beta)^{-q}\left(1 + \frac{a}{\alpha}\right)^{q-2}\left(1 - \frac{a}{a_f}\right)^{(2-q)/q}, \tag{6.4}$$

and from (4.12)

$$t = \beta^2(\omega\beta)^{-q}\int_0^a \left(1 + \frac{a}{\alpha}\right)^{q-1}\left(1 - \frac{a}{a_f}\right)^{(1-q)/q} \mathrm{d}a. \tag{6.5}$$

7. SOLUTION FOR $\omega\beta \ll 1$

The condition $\omega\beta \ll 1$ requires low impact velocity ($\beta \ll 1$) or slight strain-hardening ($\omega \ll 1$) or both. If $\omega\beta \ll 1$, and q^{-1} is of order one, then from (6.2) $a_f \ll v_f$. Since $v_f = \alpha/(1+\alpha)$, this implies both of the inequalities

$$a \leq a_f \ll 1, \quad a \leq a_f \ll \alpha.$$

Thus to a first approximation we may now neglect a compared to α in the results of Section 6. This gives, from (6.3), (6.4) and (6.5),

$$\left(1 - \frac{v_2}{v_f}\right)^q = 1 - \frac{a}{a_f}, \quad \left(\frac{v_1}{v_f} - 1\right)^q = \alpha^{-q}\left(1 - \frac{a}{a_f}\right), \tag{7.1}$$

$$f(a) = -\beta^2(\omega\beta)^{-q}\left(1 - \frac{a}{a_f}\right)^{(2-q)/q}, \tag{7.2}$$

$$t = \beta^2(\omega\beta)^{-q}\int_0^a \left(1 - \frac{a}{a_f}\right)^{(1-q)/q} \mathrm{d}a = t_f\left\{1 - \left(1 - \frac{a}{a_f}\right)^{1/q}\right\}, \tag{7.3}$$

where

$$t_f = qa_f\beta^2(\omega\beta)^{-q} = \beta^2 v_f, \tag{7.4}$$

gives the time at which the discontinuity ceases to propagate. Then (7.3) may be written

$$\left(1-\frac{a}{a_f}\right) = \left(1-\frac{t}{t_f}\right)^q, \tag{7.5}$$

and (7.1) become

$$\frac{v_2}{v_f} = \frac{t}{t_f}, \quad \frac{v_1}{v_f} = 1 - \alpha^{-1}\left(1-\frac{t}{t_f}\right). \tag{7.6}$$

Also, from (4.13), (7.2) and (7.5), the deflection of the beam relative to its end points for $0 \leq x \leq a(t)$ is given by

$$\begin{aligned} Lu(x,t) &= \frac{1}{2}L\beta^2 v_f\left\{\left(1-\frac{x}{a_f}\right)^{2/q} - \left(1-\frac{a}{a_f}\right)^{2/q}\right\} \\ &= \frac{1}{2}L\beta^2 v_f\left\{\left(1-\frac{x}{a_f}\right)^{2/q} - \left(1-\frac{t}{t_f}\right)^{2}\right\}, \end{aligned}$$

and the final deflection, relative to the end points, is, for $0 \leq x \leq a_f$,

$$Lu(x,t_f) = \frac{1}{2}L\beta^2 v_f\left(1-\frac{x}{a_f}\right)^{2/q}. \tag{7.7}$$

The approximate solutions of Sections 6 and 7 are not valid if q (or equivalently n) is very small compared to one. In fact, it can be shown from (4.5) and (4.10) that $a_f \to 1$ as $q \to 0$, and so the assumption $a \ll 1$ used in (6.1) is invalid for small values of q.

It is interesting to note that the maximum final deflection, which is obtained by setting $x = 0$ in (7.7), is independent of the work-hardening parameter q.

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